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ORIGINAL PAPER

Automorphisms of the mapping class group of a nonorientable surface

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Abstract Let S be a nonorientable surface of genus $g \geq 5$ with $n \geq 0$ punctures, and $\mathrm{Mod}(S)$ its mapping class group. We define the complexity of S to be the maximum rank of a free abelian subgroup of $\mathrm{Mod}(S)$. Suppose that S_1 and S_2 are two such surfaces of the same complexity. We prove that every isomorphism $\mathrm{Mod}(S_1) \to \mathrm{Mod}(S_2)$ is induced by a diffeomorphism $S_1 \to S_2$. This is an analogue of Ivanov's theorem on automorphisms of the mapping class groups of an orientable surface, and also an extension and improvement of the first author's previous result.

Keywords Nonorientable surface · Mapping class group · Outer automorphism

Mathematics Subject Classification (2010) 20F38 · 57N05

1 Introduction

Let $\Sigma^n_{g,b}$ (resp. $N^n_{g,b}$) denote the orientable (resp. nonorientable) surface of genus g with b boundary components and n punctures (or distinguished points). If b or n equals 0, then we drop it from the notation. Let $\mathrm{Mod}(N^n_{g,b})$ denote the mapping class group of $N^n_{g,b}$, which is the group of isotopy classes of all diffeomorphisms of $N^n_{g,b}$, where diffeomorphisms and isotopies are the identity on the boundary. The mapping class group $\mathrm{Mod}(\Sigma^n_{g,b})$ is

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defined analogously, but we consider only orientation preserving maps. The pure mapping class groups $\mathrm{PMod}(\Sigma^n_{g,b})$ and $\mathrm{PMod}(N^n_{g,b})$ are the subgroups of $\mathrm{Mod}(\Sigma^n_{g,b})$ and $\mathrm{Mod}(N^n_{g,b})$ respectively, consisting of the isotopy classes of diffeomorphisms fixing each puncture. We denote by $\mathrm{PMod}^+(N^n_{g,b})$ the subgroup of $\mathrm{PMod}(N^n_{g,b})$ consisting of the isotopy classes of diffeomorphisms preserving local orientation at each puncture. Finally, let $\mathcal{T}(N^n_{g,b})$ denote the twist subgroup of $\mathrm{PMod}^+(N^n_{g,b})$ generated by Dehn twists about all two-sided curves.

We define the *complexity* of N^n_g, denoted by $\xi\left(N^n_g\right)$, as the maximum rank of a free abelian subgroup of $\mathrm{Mod}(N^n_g)$. By [8], for $g+n>2$ we have

$$\xi\left(N^n_g\right)=\begin{cases}\frac{3}{2}(g-1)+n-2 & \text{if } g \text{ is odd}\\ \frac{3}{2}g+n-3 & \text{if } g \text{ is even.}\end{cases}$$

The first author proved in [2] that the outer automorphism group of $\mathrm{Mod}(N_g)$ is cyclic for $g\geq 5$. In this paper we improve this result and also extend it to the case of surfaces with punctures.

Theorem 1.1 *For $i=1,2$ let $S_i=N^{n_i}_{g_i}$ be a nonorientable surface of genus $g_i\geq 5$ with $n_i\geq 0$ punctures, and assume $\xi(S_1)=\xi(S_2)$. Then every isomorphism $\mathrm{Mod}(S_1)\to\mathrm{Mod}(S_2)$ is induced by a diffeomorphism $S_1\to S_2$.*

In particular, for $S_1=S_2$ we obtain the following.

Corollary 1.2 *The outer automorphism group $\mathrm{Out}(\mathrm{Mod}(N^n_g))$ is trivial for $g\geq 5$ and $n\geq 0$.*

The analogous theorem for the mapping class group of an orientable surface is due to Ivanov [5], who proved that if Σ is an orientable surface of genus $g\geq 3$, then every automorphism of $\mathrm{Mod}(\Sigma)$ is induced by a diffeomorphism of Σ, not necessarily orientation preserving. Later, Ivanov and McCarthy [6] proved (among other things) that any injective endomorphism of $\mathrm{Mod}(\Sigma)$ must be an isomorphism. Finally, by recent results of Castel [4] and Aramayona-Souto [1], any nontrivial endomorphism of $\mathrm{Mod}(\Sigma)$ must be an isomorphism. It seems reasonable to expect that Theorem 1.1 is true also for surfaces of genus less than 5 and sufficiently big complexity. On the other hand, Corollary 1.2 does not hold for $(g,n)=(2,0)$ or $(3,1)$, see [2, Proposition 4.5].

Similarly as in [5,6], the main ingredient of our proof of Theorem 1.1 is an algebraic characterization of Dehn twists (Theorem 2.4), from which we conclude that any isomorphism $\mathrm{Mod}(S_1)\to\mathrm{Mod}(S_2)$ maps Dehn twists on Dehn twists. However, unlike for orientable surfaces, $\mathrm{Mod}(N^n_g)$ is not generated by Dehn twists (and neither are $\mathrm{PMod}(N^n_g)$ and $\mathrm{PMod}^+(N^n_g)$, see [7,14]). In Subsection 2.8 we fix a finite generating set of $\mathrm{PMod}^+(N^n_g)$ consisting of Dehn twists and one crosscap transposition. By using this generating set we show that any isomorphism $\mathrm{Mod}(S_1)\to\mathrm{Mod}(S_2)$ restricts to an isomorphism $\mathrm{PMod}^+(S_1)\to\mathrm{PMod}^+(S_2)$ of the form $x\mapsto fxf^{-1}$ for some diffeomorphism $f\colon S_1\to S_2$. Then we conclude Theorem 1.1 by using the following lemma proved in [5].

Lemma 1.3 *(Ivanov) Let H be a normal subgroup of a group G such that the centralizer of H in G is trivial. If $\varphi\colon G\to G$ is an automorphism such that $\varphi(x)=x$ for all $x\in H$, then $\varphi=\mathrm{id}_G$.*

We close this introduction by remarking that Corollary 1.2 together with the fact that the center of $\mathrm{Mod}(N^n_g)$ is trivial [13, Corollary 6.3], imply that $\mathrm{Aut}(\mathrm{Mod}(N^n_g))$ is isomorphic to $\mathrm{Mod}(N^n_g)$ for $g\geq 5$.

2 Preliminaries

Let G be a group, $X \subseteq G$ a subset and $x \in G$ an element of G. Then $C(G)$, $C_G(X)$ and $C_G(x)$ will denote the center of G, the centralizer of X in G and the centralizer of x in G, respectively.

Let $g = 2\rho + m$ for $\rho \geq 0, m \geq 1$. We can represent N_g^n as an orientable surface of genus ρ with n punctures and m crosscaps. In the figures, a crosscap is drawn as a disc with a cross (e.g. Fig. 1). This means that the interior of the disc should be removed from the surface, and then antipodal points on the resulting boundary component should be identified.

2.1 Curves and Dehn twists

By a *curve* a on a surface S we understand in this paper an unoriented simple closed curve. According to whether a regular neighbourhood of a is an annulus or a Möbius strip, we call a two-sided or one-sided respectively. If a bounds a disc with at most one puncture or a Möbius band, then it is called trivial. Otherwise, we say that it is nontrivial. Let S^a denote the surface obtained by cutting S along a. If S^a is connected, then we say that a is nonseparating. Otherwise, a is called separating. If a is two-sided, then we denote by t_a a Dehn twist about a. On a nonorientable surface it is impossible to distinguish between right- and left-handed twists, so the direction of a twist t_a has to be specified for each curve a. Equivalently we may choose an orientation of a regular neighbourhood of a. Then t_a denotes the right-handed Dehn twist with respect to the chosen orientation. Unless we specify which of the two twists we mean, t_a denotes any of the two possible twists. It is proved in [13] that many well known properties of Dehn twists on orientable surfaces are also satisfied in the nonorientable case. We will use these properties in this paper.

For two curves a and b we denote by $i(a, b)$ their geometric intersection number (see [13] for definition and properties). We say that a and b are *equivalent* if there exists a diffeomorphism $h\colon S \to S$ such that $h(a) = b$.

We say that a collection of curves $\mathcal{C} = \{a_1, \ldots, a_k\}$ is a *multicurve* if the curves a_i are nontrivial, pairwise disjoint, pairwise nonisotopic, and none is isotopic to a boundary component of S. We denote by $S^{\mathcal{C}}$ the surface obtained by cutting S along all curves of $\mathcal{C}$.

2.2 Pants and skirts

We will use some properties of pants and skirts (P-S) decompositions defined in [13, Section 5]. We say that a multicurve $\mathcal{C}$ is a P-S decomposition if each $a \in \mathcal{C}$ is two-sided and each component of $S^{\mathcal{C}}$ is diffeomorphic to one of the following surfaces:

- disc with 2 punctures (pair of pants of type 1),
- annulus with 1 puncture (pair of pants of type 2),
- sphere with 3 holes (pair of pants of type 3),
- Möbius strip with 1 puncture (skirt of type 1),
- Möbius strip with 1 hole (skirt of type 2).

A P-S decomposition $\mathcal{C}$ is called *separating* if each $a \in \mathcal{C}$ is a boundary of two different connected components of $S^{\mathcal{C}}$.

Lemma 2.1 *Let $S = N_g^n$ for $g \geq 3$, $s = \xi(S)$ if $g \neq 4$, and $s = 2 + n$ if $g = 4$. Suppose that a is a two-sided curve on S. There exits a P-S decomposition $\mathcal{C} = \{a_1, \ldots, a_s\}$ of S, such that each a_i is equivalent to a, if and only if S^a is connected and nonorientable. Furthermore, if $g + n > 3$ then such P-S decomposition must be separating.*

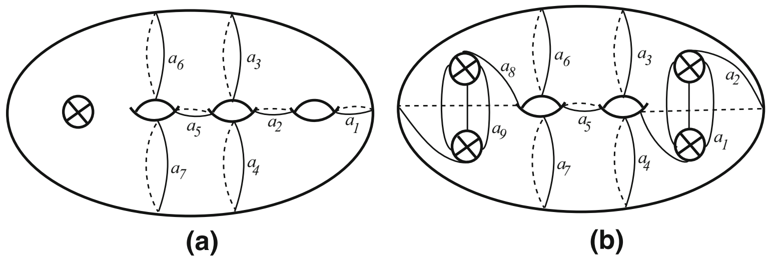


Fig. 1 P-S decomposition of **a** N_7 and **b** N_8 satisfying the conditions of Lemma 2.1

Proof The "if" part is left to the reader (see Fig. 1). Suppose that a is separating. Then all a_i are separating. Furthermore, either each a_i separates a pair of pants of type 1, or each a_i separates a skirt of type 1. It follows that $s \leq n$, a contradiction. Now suppose that S^{a_i} is connected and orientable (this is possible only for even g). Then every component of $S^{\mathcal{C}}$ is a pair of pants of type either 2 or 3. Note, however, that for $i \neq j$ the curves a_i, a_j together separate S (there can be no curve on S disjoint from a_i and intersecting a_j once; such a curve would be two-sided and one-sided at the same time). It follows that no component of $S^{\mathcal{C}}$ is a pair of pants of type 3, hence all components are pairs of pants of type 2. We have $s \leq n$, a contradiction.

Now suppose that $g + n > 3$ and a_i is a boundary of only one connected component P of $S^{\mathcal{C}}$. Because $-\chi(S) = g + n - 2 > 1$, $S^{\mathcal{C}}$ has more than one component. It follows that P must be a pair of pants of type 3 and the third boundary component of P must separate S. This is a contradiction, because all a_i are nonseparating. Thus $\mathcal{C}$ is a separating P-S decomposition of S. □

Remark 2.2 For $g = 4$ there also exist P-S decompositions of N_4^n of cardinality $\xi(N_4^n) = 3 + n$. However, such a P-S decomposition can not consist of nonseparating curves with nonorientable component.

Lemma 2.3 *Let $S = N_g^n$ for $g \geq 3$, $(g, n) \notin \{(3, 0), (4, 0)\}$ and suppose that $\mathcal{C} = \{a_1, \ldots, a_s\}$ is a P-S decomposition as in Lemma 2.1, where $s = \xi(S)$ if $g \neq 4$, and $s = 2 + n$ if $g = 4$. For $k \geq 1$ let $T_{\mathcal{C}}^k$ be the subgroup of* $\mathrm{Mod}(S)$ *generated by $t_{a_i}^k$ for $1 \leq i \leq s$. Then, for each $k \geq 1$:*

(a) $T_{\mathcal{C}}^k$ is a free abelian group of rank s;
(b) $C_{\mathrm{Mod}(S)}(T_{\mathcal{C}}^k) = T_{\mathcal{C}}^1$.

Proof The assertion (a) follows from [13, Proposition 4.4]. To prove (b) we use an idea from the proof of [13, Theorem 6.2]. Suppose $f \in C_{\mathrm{Mod}(S)}(T_{\mathcal{C}}^k)$. Then $t_{a_i}^k = f t_{a_i}^k f^{-1} = t_{f(a_i)}^k$ for all i. It follows that f fixes each curve a_i, hence it permutes the connected components of $S^{\mathcal{C}}$. Suppose that f interchanges some two components P_1 and P_2 of $S^{\mathcal{C}}$. By the proof of Lemma 2.1, there are no pairs of pants of type 1 and no skirts of type 1 in the decomposition. Suppose that P_1 and P_2 are skirts of type 2 glued along a curve a_i. Then the remaining boundary curves $a_j \subset P_1$ and $a_l \subset P_2$ must be glued together ($a_l = f(a_j) = a_j$), hence S is the closed nonorientable surface of genus 4, contrary to the assumptions of the lemma. Similarly, if P_1 and P_2 are pairs of pants of type 2 or 3, then S must be a Klein bottle with two punctures, or a closed nonorientable surface of genus 4 respectively, which again contradicts the assumptions. Thus f fixes each component of $S^{\mathcal{C}}$. Furthermore, since f centralizes the boundary twists of each pair of pants, it preserves its orientation. Because the mapping class groups of a pair of pants of type 2 or 3, and that of the skirt of type 2 are generated by boundary twists, f is a product of some powers of t_{a_i} for $1 \le i \le s$. Thus $C_{\mathrm{Mod}(S)}(T^k_{\mathcal{C}}) \subseteq T^1_{\mathcal{C}}$ and the opposite inclusion is obvious. □

Note that (b) of Lemma 2.3 implies that $T^1_{\mathcal{C}}$ is a maximal abelian subgroup of $\mathrm{Mod}(S)$.

2.3 Pure subgroups

Let S denote the surface N^n_g for $g \ge 3$ and $n \ge 0$. We recall from [2] the construction of finite index pure subgroups $\Gamma_m(S)$ of $\mathrm{Mod}(S)$ (see Section 2 of [2] for more details). Fix an orientable double cover $\Sigma = \Sigma^{2n}_{g-1}$ of S. Then $\mathrm{Mod}(S)$ can be identified with the subgroup of $\mathrm{Mod}(\Sigma)$, consisting of the isotopy classes of diffeomorphisms commuting with the covering involution. Consequently, $\mathrm{Mod}(S)$ acts on $H_1(\Sigma, \mathbb{Z}/m\mathbb{Z})$ for all $m \ge 0$. We define $\Gamma_m(S)$ to be the subgroup of $\mathrm{Mod}(S)$ consisting of all elements inducing the identity on $H_1(\Sigma, \mathbb{Z}/m\mathbb{Z})$. If $m \ge 3$, then $\Gamma_m(S)$ is a pure subgroup of $\mathrm{Mod}(S)$. In particular, if $f \in \Gamma_m(S)$ preserves a multicurve $\mathcal{C}$, then f fixes each curve of $\mathcal{C}$ and, furthermore, it can be represented by a diffeomorphism equal to the identity on a regular neighbourhood of each curve of $\mathcal{C}$. If the restriction of f to any connected component of $S^{\mathcal{C}}$ is isotopic (by an isotopy that does not have to fix pointwise the boundary components of $S^{\mathcal{C}}$) either to the identity or to a pseudo-Anosov map, then $\mathcal{C}$ is called a *reduction system* for f. The intersection of all reduction systems for f is called the *canonical reduction system* for f. Reduction systems were introduced by Birman, Lubotzky and McCarthy in [3], for the case of a nonorientable surface see [16].

2.4 Algebraic characterization of a Dehn twist

The key ingredient of the proof of our main result is an algebraic characterization of a Dehn twist about a nonseparating curve in the mapping class group. Theorem 2.4 below is an extension of Theorem 3.1 of [2] to punctured surfaces. The proof closely follows Ivanov's ideas [5].

Theorem 2.4 *Let $S = N^n_g$ for $g \ge 3$, $(g, n) \notin \{(3, 0), (4, 0)\}$ and let Γ be a finite index subgroup of $\Gamma_m(S)$ for $m \ge 3$. An element $f \in \mathrm{Mod}(S)$ is a Dehn twist about a nonseparating curve with nonorientable complement if and only if the following conditions are satisfied:*

(i) $C(C_\Gamma(f^k)) \cong \mathbb{Z}$, *for any integer $k \ne 0$ such that $f^k \in \Gamma$.*
(ii) *Set $s = \xi(S)$ if $g \ne 4$, and $s = 2 + n$ if $g = 4$. There exist elements $f_2, \ldots, f_s \in$* $\mathrm{Mod}(S)$*, each conjugate to $f_1 = f$, such that $f_1, \ldots, f_s$ generate a free abelian group K of rank s.*
(iii) *For $k \ge 1$ let K_k be the subgroup of* $\mathrm{Mod}(S)$ *generated by f^k_i for $1 \le i \le s$. Then* $C_{\mathrm{Mod}(S)}(K_k) = K$.

Proof Assume that the above conditions are satisfied, then we have to show that f is a Dehn twist about a nonseparating curve with nonorientable complement.

Choose any integer $k \ne 0$ such that $f^k \in \Gamma$. Because f has infinite order by (ii), f^k is not the identity element. Let $\mathcal{C}$ be the canonical reduction system for f^k. Let G denote the subgroup generated by the twists about the two-sided curves in $\mathcal{C}$. Set $G' = G \cap \Gamma$. Then G and G' are free abelian groups. Firstly, we will show that $G' \subset C(C_\Gamma(f^k))$. Let $g \in C_\Gamma(f^k)$. Since g commutes with f^k, it preserves the canonical reduction system $\mathcal{C}$. Because g is pure, it fixes each curve of $\mathcal{C}$ and also preserves orientation of a regular neighbourhood of each two-sided curve of $\mathcal{C}$. It follows that g commutes with each generator G, hence $G \subseteq C_{\mathrm{M}(S)}(C_\Gamma(f^k))$. So, $G' \subset C_\Gamma(C_\Gamma(f^k)) = C(C_\Gamma(f^k))$. For the last equality observe that, since $f^k \in C_\Gamma(f^k)$, $C_\Gamma(C_\Gamma(f^k)) \subseteq C_\Gamma(f^k)$, hence $C_\Gamma(C_\Gamma(f^k)) \subseteq C(C_\Gamma(f^k))$ and the opposite inclusion is obvious. The assumption $C(C_\Gamma(f^k)) = \mathbb{Z}$ implies that $\mathcal{C}$ contains at most one two-sided curve.

Assume that $\mathcal{C}$ has no two-sided curve, so that $\mathcal{C} = \{c_1, \ldots, c_l\}$, where each c_i is a one-sided curve. Then $S^{\mathcal{C}}$ is connected. Let $\mathrm{Stab}^+(\mathcal{C})$ be the subgroup of $\mathrm{Mod}(S)$ consisting of elements fixing each curve of $\mathcal{C}$ and preserving its orientation. Note that $C_\Gamma(f^k) \subseteq \mathrm{Stab}^+(\mathcal{C})$. The mapping $h \mapsto h_{|S^{\mathcal{C}}}$ defines an isomorphism $\mathrm{Stab}^+(\mathcal{C}) \to \mathrm{Mod}(S^{\mathcal{C}})/\mathbb{Z}^l$, where $\mathbb{Z}^l$ is the subgroup generated by the boundary twists of $S^{\mathcal{C}}$ (see [12, Section 4]). We also have a monomorphism $\mathrm{Mod}(S^{\mathcal{C}})/\mathbb{Z}^l \to \mathrm{Mod}(S')$, where S' is the surface obtained from $S^{\mathcal{C}}$ by collapsing each boundary component to a puncture. By composing these two maps we obtain a monomorphism $\theta\colon \mathrm{Stab}^+(\mathcal{C}) \to \mathrm{Mod}(S')$. Because $\mathcal{C}$ is the canonical reduction system for f^k, $\theta(f^k)$ is either the identity or pseudo-Anosov. In the former case f^k must be the identity, a contradiction. Suppose $\theta(f^k)$ is pseudo-Anosov. Set $H = \Gamma \cap K_k$, where K_k is the group from condition (iii). We have $H \subseteq C_\Gamma(f^k) \subseteq \mathrm{Stab}^+(\mathcal{C})$ and $\theta(H)$ is a free abelian subgroup of $\mathrm{Mod}(S')$ containing $\theta(f^k)$. Since $\theta(f^k)$ is pseudo-Anosov, $\theta(H)$ must have rank 1. This is a contradiction, as H has rank $s > 1$.

We have $\mathcal{C} = \{c_1, \ldots, c_l, a\}$, where a is a two-sided curve and each c_i is one-sided. Let D be the subgroup generated by f^k and the twist about a and denote the intersection $D \cap \Gamma$ by D'. Hence, $D' \subset C(C_\Gamma(f^k))$ and hence D' is isomorphic to $\mathbb{Z}$. It follows that $f^{k_1} = t_a^m$ for some integers m and k_1 (possibly greater than k).

Let $f_1, \ldots, f_s$ be the elements from condition (ii). For $1 \leq i \leq s$ we have $f_i^{k_1} = t_{a_i}^m$ for some curve a_i equivalent to $a_1 = a$. We claim that $\mathcal{C} = \{a_1, \ldots, a_s\}$ is a P-S decomposition of S. If not, then we can complete $\mathcal{C}$ to a P-S decomposition $\mathcal{C}'$. Let $T_{\mathcal{C}'}$ be the free abelian group generated by twists about the curves of $\mathcal{C}'$. We have $T_{\mathcal{C}'} \subseteq C_{\mathrm{Mod}(S)}(K_{k_1}) = K$. It follows that $\mathrm{rank}(T_{\mathcal{C}'}) \leq s$, hence $\mathcal{C}' = \mathcal{C}$. By (iii) and (b) of Lemma 2.3 we have $K = C_{\mathrm{Mod}(S)}(K_{k_1}) = C_{\mathrm{Mod}(S)}(T_{\mathcal{C}}^m) = T_{\mathcal{C}}^1$. By (ii) f is a primitive element of $K = T_{\mathcal{C}}^1$, hence $f = t_{a_1}$. It follows from Lemma 2.1 that a_1 is nonseparating and has nonorientable complement.

The proof of the opposite implication is straightforward and left to the reader (see [5]). □

Corollary 2.5 *For $i = 1, 2$ let $S_i = N_{g_i}^{n_i}$ for $g_i \geq 5$ and assume $\xi(S_1) = \xi(S_2)$. Suppose that $\varphi\colon \mathrm{Mod}(S_1) \to \mathrm{Mod}(S_2)$ is an isomorphism. If $f \in \mathrm{Mod}(S_1)$ is a Dehn twists about a nonseparating curve with nonorientable complement, then so is $\varphi(f)$.*

Proof Fix $m \geq 3$. Because f satisfies the conditions (i), (ii), (iii) of Theorem 2.4 with $\Gamma_m(S_1)$ as Γ, it follows that $\varphi(f)$ also satisfies (i), (ii), (iii) of Theorem 2.4 with $\Gamma = \varphi(\Gamma_m(S_1)) \cap \Gamma_m(S_2)$. □

2.5 Chains

A sequence $(a_1, \ldots, a_k)$ of curves is called a chain if $i(a_i, a_{i+1}) = 1$ for $1 \leq i \leq k-1$ and $i(a_i, a_j) = 0$ for $|i-j| > 1$. The integer $k \geq 1$ is called the length of the chain. If all curves in a chain are two-sided, then a regular neighbourhood of the union of these curves is orientable. Let t_{a_i} be right-handed Dehn twists with respect to some orientation of such a neighbourhood for $1 \leq i \leq k$. Then

(a) $t_{a_i} t_{a_{i+1}} t_{a_i} = t_{a_{i+1}} t_{a_i} t_{a_{i+1}}$ for $1 \leq i \leq k-1$
(b) $t_{a_i} t_{a_j} = t_{a_j} t_{a_i}$ for $|i-j| > 1$.

Conversely, if a sequence of Dehn twists $(t_{a_1}, \dots, t_{a_k})$ satisfies (a) and (b), then $(a_1, \dots, a_k)$ is a chain, and the twists are right-handed with respect to some orientation of a regular neighbourhood of the union of the curves of the chain (see [13, Section 4]). A sequence of Dehn twists satisfying (a) and (b) will also be called a chain. Observe that if (a_1, a_2) is a 2-chain of two-sided curves, then S^{a_i} must be connected and nonorientable for $i = 1, 2$.

2.6 Trees

We will now define a tree of curves (and Dehn twists) as a generalization of a chain. Suppose that $\mathcal{C}$ is a collection of curves, such that $i(a, b) \in \{0, 1\}$ for all $a, b \in \mathcal{C}$. Let $\Gamma(\mathcal{C})$ be a graph with $\mathcal{C}$ as the set of vertices, and where a and b are connected by an edge if and only if $i(a, b) = 1$. We will call $\mathcal{C}$ a tree if and only if $\Gamma(\mathcal{C})$ is a tree (connected and acyclic). If all curves in a tree are two-sided, then a regular neighbourhood of the union of these curves is orientable. Let t_a be right-handed Dehn twists with respect to some orientation of such a neighbourhood for $a \in \mathcal{C}$. Then

(a′) $t_a t_b t_a = t_b t_a t_b$ if a and b are connected by an edge,
(b′) $t_a t_b = t_b t_a$ otherwise.

Conversely, suppose that $T = \{t_a : a \in \mathcal{C}\}$ is a set of Dehn twists for some set of curves $\mathcal{C}$, where each two twists of T either commute, or satisfy the braid relation. Then the geometric intersection number of the underlying curves is, respectively, either 0 or 1. We say that T is a tree of twists if and only if $\mathcal{C}$ is a tree. We will always assume that the curves in $\mathcal{C}$ realize their geometric intersection number and a regular neighbourhood of the union of these curves is oriented so that all twists of T are right-handed.

The following corollary follows immediately from Corollary 2.5

Corollary 2.6 *For $i = 1, 2$ let $S_i = N_{g_i}^{n_i}$ for $g_i \geq 5$ and assume $\xi(S_1) = \xi(S_2)$. Suppose that $\varphi\colon \mathrm{Mod}(S_1) \to \mathrm{Mod}(S_2)$ is an isomorphism. If $T = \{t_a : a \in \mathcal{C}\} \subset \mathrm{Mod}(S_1)$ is a tree of Dehn twists of cardinality at least 2, then $\varphi(T)$ is also a tree of Dehn twists for some set of curves $\mathcal{C}'$, such that $\Gamma(\mathcal{C})$ and $\Gamma(\mathcal{C}')$ are isomorphic (as abstract graphs).*

2.7 Useful relations among Dehn twists

The following lemma is well-known (see [9, Proposition 2.12]).

Lemma 2.7 *Suppose that $(t_{c_1}, t_{c_2}, \dots, t_{c_{2k+1}})$ is a chain of twists. Then*

$$(t_{c_1} t_{c_2} \dots t_{c_{2k+1}})^{2k+2} = t_{u_1} t_{u_2},$$

where t_{u_1}, t_{u_2} are right-handed twists about the boundary components of a regular neighbourhood of the union of the curves c_i (Fig. 2).

Relations (a) and (b) of the next lemma appear in [9, Theorem 3.2] as (R5) and (R6) respectively. Their proof can be deduced from [9, Proposition 2.12].

Lemma 2.8 *Suppose that $\{t_{c_0}, t_{c_1}, \dots, t_{c_7}\}$ is the tree of right-handed Dehn twists on $\Sigma_{2,3}$ whose underlying curves are shown on Fig. 3, and t_{u_i}, $i = 1, 2, 3$, are right-handed Dehn twists about the boundary components of $\Sigma_{2,3}$. Then*

(a) $t_{u_1} = (t_{c_0} t_{c_1} t_{c_2} t_{c_3} t_{c_4} t_{c_5})^5 (t_{c_1} t_{c_2} t_{c_3} t_{c_4} t_{c_5})^{-6}$

(b) $t_{u_2} = (t_{c_7} t_{c_6} t_{c_4} t_{c_3} t_{c_2} t_{c_0})^5 (t_{c_6} t_{c_4} t_{c_3} t_{c_2} t_{c_0})^{-6} (t_{c_6} t_{c_5} t_{c_4})^4 (t_{c_7} t_{c_6} t_{c_5} t_{c_4})^{-3}$

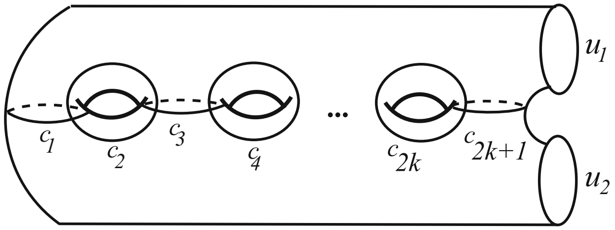
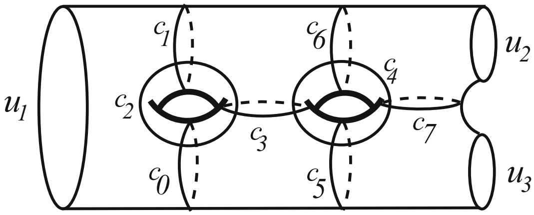


Fig. 2 A chain of two-sided curves of odd length and its regular neighbourhood

Fig. 3 The curves from Lemma 2.8



2.8 Generators of $\mathrm{PMod}^+(N_h^n)$

The aim of this subsection is to fix a finite generating set of $\mathrm{PMod}^+(N_h^n)$ for $h \geq 5$. We choose a generating set which differs slightly from the one given in [14, Theorem 4.1]. We begin its description with Dehn twists. Let D and E be the trees of curves from Figs. 4 and 5. We will abuse notation and denote by the same symbols the corresponding trees of Dehn twists. As we already mentioned in the introduction, $\mathrm{PMod}^+(N_h^n)$ is not generated by Dehn twists and to obtain a generating set for this group we add to D or E one more generator, namely a crosscap transposition (in [14] a crosscap slide is used). In order to describe this element, and also to be able to prove Lemmas 3.7 and 3.8 in Sect. 3, we view certain subsurface of N_h^n as a disc with crosscaps.

For $k \in \{5, 6\}$ let $N_{k,1}$ be a nonorientable surface of genus k with one boundary component, represented on Fig. 6 as disc with k crosscaps numbered from 1 to k. For $i \leq j$ let $c_{i,j}$ denote the simple closed curve on $N_{k,1}$ from Fig. 6. Note that $c_{i,j}$ is two-sided if and only if $j - i$ is odd. In such case $t_{c_{i,j}}$ denotes the twist about $c_{i,j}$ in the direction indicated by the arrows on Fig. 6.

We denote by u the *crosscap transposition* defined to be the isotopy class of the diffeomorphism of $N_{k,1}$ interchanging the $(k-1)$'st and k'th crosscaps as shown on Fig. 7, and equal to the identity outside a disc containing these crosscaps.

Lemma 2.9 *For $g \geq 2$ there are embeddings $\theta_1 \colon N_{5,1} \to N_{2g+1}^n$ and $\theta_2 \colon N_{6,1} \to N_{2g+2}^n$, such that:*

(a) for $i = 1, 2$, $N_{2g+i}^n \backslash \theta_i(N_{4+i,1})$ is an orientable surface of genus $g - 2$ with n punctures containing the curves a_i for all $i > 8$;
(b) for $i = 1, 2$, $a_5 = \theta_i(c_{1,2})$, $a_6 = \theta_i(c_{2,3})$, $a_4 = \theta_i(c_{3,4})$, $a_2 = \theta_i(c_{4,5})$, $a_1 = \theta_i(c_{1,4})$;
(c) $a_3 = \theta_1(t_{c_{4,5}} u^{-1}(c_{1,4}))$;
(d) $a_0 = \theta_2(c_{5,6})$, $a = \theta_2(c_{1,6})$;
(e) θ_2 maps boundary curves of a regular neighbourhood of $c_{1,6} \cup c_{5,6} \cup c_{6,6}$ on a_1 and a_3.

Proof Suppose $h = 2g + 1$. Set $c_5 = c_{1,2}$, $c_6 = c_{2,3}$, $c_4 = c_{3,4}$, $c_2 = c_{4,5}$, $c_1 = c_{1,4}$ and $c_3 = t_{c_{4,5}} u^{-1}(c_{1,4})$. By changing these curves by a small isotopy, we may assume that

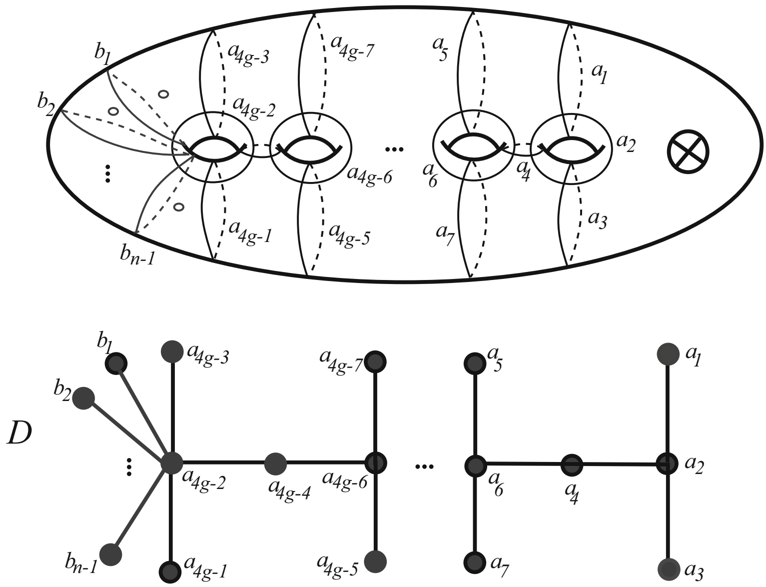


Fig. 4 The tree of curves D on N_{2g+1}^n

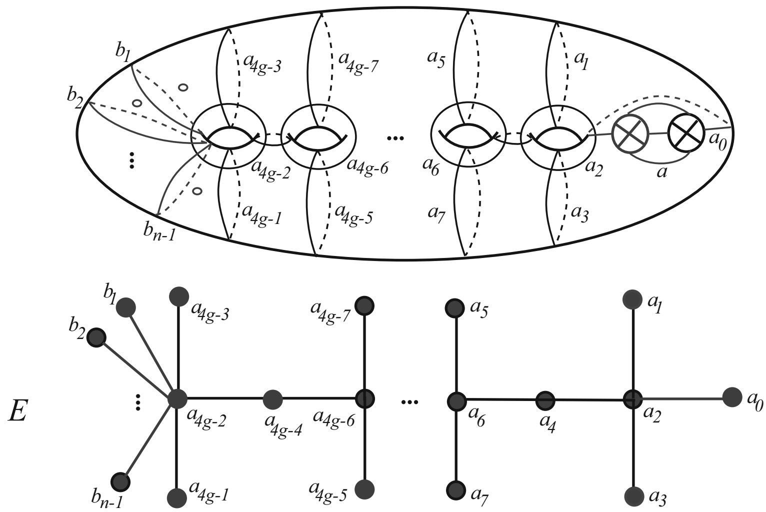


Fig. 5 The tree of curves E on N_{2g+2}^n

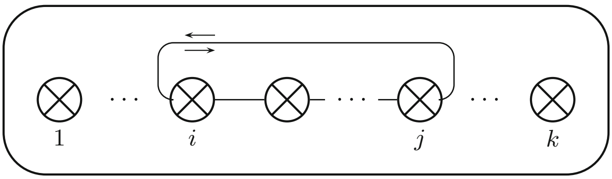


Fig. 6 The surface $N_{k,1}$ and the curve $c_{i,j}$, $k = 5$ or 6

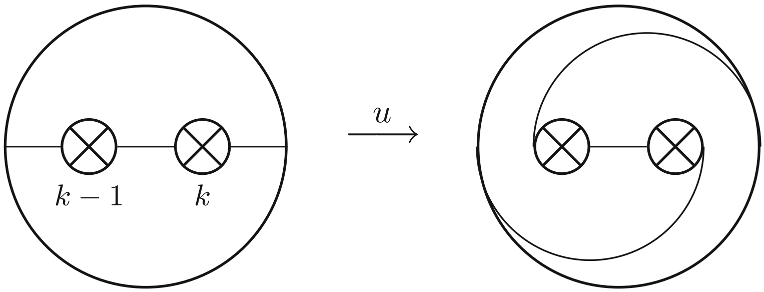


Fig. 7 The crosscap transposition

they realize their geometric intersection number. Then we have $|c_i \cap c_j| = |a_i \cap a_j|$ for all $i, j \in \{1, \dots, 6\}$. Let M (resp. M') be a regular neighbourhood of the union of c_i (resp. a_i) for $i \in \{1, \dots, 6\}$. Observe that M and M' are both diffeomorphic to $\Sigma_{2,3}$. There is a diffeomorphism $\theta' \colon M \to M'$ such that $\theta'(c_i) = a_i$ for $i \in \{1, \dots, 6\}$. To see that θ' can be extended to an embedding $\theta_1 \colon N_{5,1} \to N_h^n$ observe that (1) c_1, c_4 and c_5 (resp. a_1, a_4 and a_5) bound a pair of pants on $N_{5,1}$ (resp. N_h^n); (2) c_3, c_4, c_5 and $\partial N_{5,1}$ bound a 4-holed sphere; (3) c_1 and c_3 (resp. a_1 and a_3) bound a subsurface of $N_{5,1}$ (resp. N_h^n) diffeomorphic to $N_{1,2}$.

Suppose $h = 2g + 2$. Set $c_5 = c_{1,2}, c_6 = c_{2,3}, c_4 = c_{3,4}, c_2 = c_{4,5}, c_1 = c_{1,4}, c_0 = c_{5,6}$. Let K be a regular neighbourhood of $c_{1,6} \cup c_{5,6} \cup c_{6,6}$. Observe that K is a Klein bottle with two holes, whose one boundary component is isotopic to $c_1 = c_{1,4}$. Let c_3 denote the other component of ∂K. We have $|c_i \cap c_j| = |a_i \cap a_j|$ for all $i, j \in \{0, \dots, 6\}$. Let M (resp. M') be a regular neighbourhood of the union of c_i (resp. a_i) for $i \in \{0, \dots, 6\}$. Observe that M and M' are both diffeomorphic to $\Sigma_{2,4}$. There is a diffeomorphism $\theta' \colon M \to M'$ such that $\theta'(c_i) = a_i$ for $i \in \{0, \dots, 6\}$. To see that θ' can be extended to an embedding $\theta_2 \colon N_{6,1} \to N_h^n$ observe that (1) c_1, c_4 and c_5 (resp. a_1, a_4 and a_5) bound a pair of pants on $N_{6,1}$ (resp. N_h^n); (2) c_3, c_4, c_5 and $\partial N_{6,1}$ bound a 4-holed sphere; (3) two boundary curves of M (resp. M') bound an annulus with core $c_{1,6}$ (resp. a). The conditions (a, b, d, e) follow immediately from the construction of θ_2. □

Via these embeddings, we will treat $N_{4+i,1}$ as a subsurface of N_{2g+i}^n for $i = 1, 2$. Consequently, we will identify curves on $N_{4+i,1}$ with their images on N_{2g+i}^n, and also, using [15, Corollary 3.8], treat $\mathrm{Mod}(N_{4+i,1})$ as a subgroup of $\mathrm{Mod}(N_{2g+i}^n)$ (in particular $t_{a_5} = t_{c_{1,2}}$ etc.).

Proposition 2.10 *For $g \geq 2$, $\mathrm{PMod}^+(N_{2g+1}^n)$ (resp. $\mathrm{PMod}^+(N_{2g+2}^n)$) is generated by u and D (resp. u and E).*

Proof Let $y = t_{c_{k-1,k}} u$. This element is called crosscap slide and Stukow proved in [14, Theorem 4.1] that $\mathrm{PMod}^+(N_{2g+1}^n)$ is generated by $D \cup \{y\} = D \cup \{t_{a_2} u\}$, whereas $\mathrm{PMod}^+(N_{2g+2}^n)$ is generated by $E \cup \{y, t_a\} = E \cup \{t_{a_0} u, t_a\}$. It suffices to show that t_a can be expressed as a product of elements of E. This can be achieved by (a) of Lemma 2.8:

$$(t_{a_0} t_{a_1} t_{a_2} t_{a_4} t_{a_6} t_{a_5})^5 (t_{a_0} t_{a_2} t_{a_4} t_{a_6} t_{a_5})^{-6} = t_a$$

□

We will also need the following fact about the twist subgroup.

Lemma 2.11 *For $h \geq 5$, $\mathcal{T}(N_h^n)$ is generated by Dehn twists about nonseparating curves with nonorientable complement.*

Proof Set $S = N_h^n$. By [14], $\mathcal{T}(S)$ is a subgroup of $\mathrm{PMod}^+(S)$ of index 2, and $\mathrm{PMod}^+(S) = \mathcal{T}(S) \cup u\mathcal{T}(S)$. By Proposition 2.10, $\mathcal{T}(S)$ is generated by $D \cup uDu^{-1} \cup \{u^2\}$ if $h = 2g+1$, and by $E \cup uEu^{-1} \cup \{u^2\}$ if $h = 2g+2$. We have $u^2 = t_e$, where e is the boundary curve of the Klein bottle with a hole shown in Fig. 7. Because D and E consist of Dehn twists about nonseparating curves with nonorientable complement, the same is true for uDu^{-1} and uEu^{-1}, and it suffices to show that t_e can also be expressed as a product of such twists. Note that S^e is homeomorphic to $N_{h-2,1}^n \amalg N_{2,1}$. If $h \geq 7$, then the surface $\Sigma_{2,3}$ from Fig. 3 can be embedded in S so that the boundary curve u_1 of $\Sigma_{2,3}$ coincides with e, and then (a) of Lemma 2.8 provides the desired expression of t_e as a product of Dehn twists about nonseparating curves with nonorientable complement.

For the case $h = 5, 6$ we need the so called star relation, which is a special case of the fourth relation of [9, Proposition 2.12]. We say that curves c_0, c_1, c_2, c_3 form a star if (c_1, c_2, c_3) is a chain, $i(c_0, c_2) = 1$ and $i(c_0, c_1) = i(c_0, c_3) = 0$. A regular neighbourhood of the union of the curves of the star is a 3-holed torus, and we denote its boundary components by u_1, u_2, u_3. The star relation is $(t_{c_0}t_{c_1}t_{c_3}t_{c_2})^3 = t_{u_1}t_{u_2}t_{u_3}$, where the twists are right-handed with respect to some orientation of the regular neighbourhood. We choose a chain (c_1, c_2, c_3) of curves such that one of the boundary components of a regular neighbourhood of $c_1 \cup c_2 \cup c_3$ is the curve e, and we denote the second component by u_1. By Lemma 2.7 we have $(t_{c_1}t_{c_2}t_{c_3})^4 = t_{u_1}t_e$. Note that the connected component of S^{u_1} containing the chain is homeomorphic to $N_{4,1}$ and so we can complete the chain to a star (c_0, c_1, c_2, c_3), by adding a curve c_0, such that one boundary curve of a regular neighbourhood of the union of the curves of the star is u_1 and the other two components bound Möbius bands. Then the star relation takes the form $(t_{c_0}t_{c_1}t_{c_3}t_{c_2})^3 = t_{u_1}$ and $t_e = (t_{c_0}t_{c_1}t_{c_3}t_{c_2})^{-3}(t_{c_1}t_{c_2}t_{c_3})^4$ is the desired expression of t_e as a product of Dehn twists about nonseparating curves with nonorientable complement. □

3 Automorphisms of $\mathrm{Mod}(N_g^n)$

The aim of this section is to prove Theorem 1.1. Our first observation is that we can assume $S_1 = S_2$ by the following lemma.

Lemma 3.1 *Suppose that* $\varphi\colon \mathrm{Mod}(S_1) \to \mathrm{Mod}(S_2)$ *is an isomorphism, where* S_1 *and* S_2 *are as in Theorem 1.1. Then* $(g_1, n_1) = (g_2, n_2)$.

Proof By Lemma 2.11 and Corollary 2.5, $\varphi(\mathcal{T}(S_1)) = \mathcal{T}(S_2)$, and hence $[\mathrm{Mod}(S_1)\colon \mathcal{T}(S_1)] = [\mathrm{Mod}(S_2)\colon \mathcal{T}(S_2)]$. Since $[\mathrm{Mod}(S_i)\colon \mathcal{T}(S_i)] = 2^{n_i+1}n_i!$ by [14, Corollary 6.4], we have $n_1 = n_2$. This and the equality $\xi(S_1) = \xi(S_2)$ imply that g_1 and g_2 must be of the same parity, and in fact $g_1 = g_2$. □

Our next goal is the following key lemma.

Lemma 3.2 *Suppose that* $h = 2g+1$ *(resp.* $h = 2g+2$*) for* $g \geq 2$ *and* $\varphi\colon \mathrm{Mod}(N_h^n) \to \mathrm{Mod}(N_h^n)$ *is an automorphism. Then there exists* $f \in \mathrm{Mod}(N_h^n)$ *such that* $\varphi(t) = ftf^{-1}$ *for each* $t \in D$ *(resp.* $t \in E \cup \{t_a\}$*).*

After we prove Lemma 3.2, the next step is to show, using Proposition 2.10, that the automorphism $\varphi'\colon \mathrm{Mod}(N_h^n) \to \mathrm{Mod}(N_h^n)$ defined as $\varphi'(x) = f^{-1}\varphi(x)f$ restricts to an inner automorphism of $\mathrm{PMod}^+(N_h^n)$. This step is completed in Lemmas 3.7 and 3.8. Finally, we conclude Theorem 1.1 by using Lemma 1.3.

For the proof of Lemma 3.2 we need to compute the centralizers of sub-trees $\Theta \subset D$ and $\Lambda \subset E$ defined as

$$\Theta = \left\{t_{a_1}, t_{a_3}, t_{a_5}\right\} \cup \left\{t_{a_{2i}} : 1 \le i \le 2g-1\right\} \cup \left\{t_{b_j} : 1 \le j \le n-1\right\},$$
$$\Lambda = \left\{t_{a_1}, t_{a_3}, t_{a_5}\right\} \cup \left\{t_{a_{2i}} : 0 \le i \le 2g-1\right\} \cup \left\{t_{b_j} : 1 \le j \le n-1\right\}.$$

Let $\Sigma_{g,n+1}$ (resp. $\Sigma_{g,n+2}$) be a subsurface of N^n_{2g+1} (resp. N^n_{2g+2}), supporting D (resp. E), obtained by removing from N^n_{2g+1} (resp. N^n_{2g+2}) n open discs, each containing one puncture, and a Möbius band (resp. an annulus with core a). For $i = 1, 2$ the inclusion $\Sigma_{g,n+i} \subset N^n_{2g+i}$ induces a homomorphism $\mathrm{Mod}(\Sigma_{g,n+i}) \to \mathrm{Mod}(N^n_{2g+i})$.

Lemma 3.3 *Suppose that $h = 2g + 1$ for $g \ge 2$. Then $C_{\mathrm{Mod}(N^n_h)}(\Theta) = 1$.*

Proof Let H denote the image of $\mathrm{Mod}(\Sigma_{g,n+1})$ in $\mathrm{Mod}(N^n_h)$. It can be easily deduced from the main result of [9] that H is generated by twists of Θ. Thus $C_{\mathrm{Mod}(N^n_h)}(\Theta) = C_{\mathrm{Mod}(N^n_h)}(H)$. Set $D' = D\backslash\{t_{a_{4i-2}} : 1 \le i \le g\}$. The curves supporting the twists of D' form a separating pants and skirts decomposition of N^n_h (see Subsection 2.2 for the definition). Let $h \in C_{\mathrm{Mod}(N^n_h)}(H)$. Since $D' \subset H$, $h \in C_{\mathrm{Mod}(N^n_h)}(D')$. By the proof of (b) of Lemma 2.3, $h = \prod t^{m_i}_{a_i}$ for some integers m_i, where the product is taken over all $t_{a_i} \in D'$. By [10, Proposition 3.4], for every $t_{a_i} \in D'$ there exists a simple closed curve c on $\Sigma_{g,n+1}$, such that $i(c, a_i) > 0$ and t_c commutes with all twists in $D'\backslash\{t_{a_i}\}$. Since $t_c \in H$, it also commutes with h. It follows that t_c commutes with $t^{m_i}_{a_i}$, which is possible only for $m_i = 0$, hence $h = 1$ and $C_{\mathrm{Mod}(N^n_h)}(H)$ is trivial. □

Lemma 3.4 *Suppose that $h = 2g + 2$ for $g \ge 2$. Then $C_{\mathrm{Mod}(N^n_h)}(\Lambda)$ is the infinite cyclic group generated by t_a, where a is the curve from Fig. 5.*

Proof Let H denote the image of $\mathrm{Mod}(\Sigma_{g,n+2})$ in $\mathrm{Mod}(N^n_h)$. Similarly as in the odd genus case, H is generated by twists of Λ, thus $C_{\mathrm{Mod}(N^n_h)}(\Lambda) = C_{\mathrm{Mod}(N^n_h)}(H)$. Note that $t_a \in H$, because a is isotopic to a boundary component of $\Sigma_{g,n+2}$. Set $E' = E \cup \{t_a\}\backslash\{t_{a_{4i-2}} : 1 \le i \le g\}$. The curves supporting the twists of E' form a separating P-S decomposition of N^n_h. Let $h \in C_{\mathrm{Mod}(N^n_h)}(H)$. By a similar argument as in the proof of (b) of Lemma 2.3, $h = t^m_a \prod t^{m_i}_{a_i}$ for some integers m_i and m, where the product is taken over all $t_{a_i} \in E'\backslash\{t_a\}$. By the same argument as in the proof for odd genus, all $m_i = 0$, hence $h = t^m_a$. □

Lemma 3.5 *Let $S = N^n_{2g+1}$ for $g \ge 2$, $n \ge 1$ and suppose that $\varphi\colon \mathrm{Mod}(S) \to \mathrm{Mod}(S)$ is an isomorphism such that $\varphi(t_{a_1})$ and $\varphi(t_{a_3})$ are Dehn twists about curves c_1 and c_3. Then $c_1 \cup c_3$ does not bound a once-punctured annulus embedded in S.*

Proof Suppose that c_1 and c_3 are the boundary curves of a once-punctured annulus embedded in S. Set $G = C_{\mathrm{Mod}(S)}\{t_{a_1}, t_{a_3}\}$ and $H = \varphi(G) = C_{\mathrm{Mod}(S)}\{t_{c_1}, t_{c_3}\}$. Observe that $S^{\{c_1,c_3\}}$ is homeomorphic to $N^{n-1}_{2g-1,2} \amalg \Sigma^1_{0,2}$, whereas $S^{\{a_1,a_3\}}$ is homeomorphic to $\Sigma^n_{g-1,2} \amalg N_{1,2}$. Let X (resp. Y) be the subsurface of S homeomorphic to $\Sigma^n_{g-1,2}$ (resp. $N^{n-1}_{2g-1,2}$) such that $\partial X = a_1 \cup a_3$ (resp. $\partial Y = c_1 \cup c_3$). The centralizer G consists of the isotopy classes of diffeomorphisms of S fixing a_1 and a_3 whose restriction to X is orientation preserving. It follows that the inclusion of X in S induces an isomorphism $\mathrm{Mod}(X) \to G$ (see [11, §5.2] or [12, §4]). Similarly, the inclusion of Y in S induces an isomorphism $\mathrm{Mod}(Y) \to H$. Let K denote the image of $\mathrm{PMod}(X)$ in G. Because $\mathrm{PMod}(X)$ is generated by Dehn twists about nonsepataing curves (see [9, Proposition 2.10]), K is generated by Dehn twists about nonseparating curves with nonorientable complement. By Corollary 2.5, $\varphi(K)$ is also generated by Dehn twists, and hence it is contained in the image of $\mathcal{T}(Y)$. By [14, Corollary 6.4], $\mathcal{T}(Y)$ has index $2^n(n-1)!$ in $\mathrm{Mod}(Y)$, and hence $[H : \varphi(K)] \geq 2^n(n-1)!$. On the other hand $[H : \varphi(K)] = [G : K] = [\mathrm{Mod}(X) : \mathrm{PMod}(X)] = n!$. This is a contradiction, because $n! < 2^n(n-1)!$. □

Proof of Lemma 3.2 Set $S = N_h^n$. Suppose $h = 2g+1$. By Corollary 2.6, $\varphi(\Theta)$ is a tree of Dehn twists for which the underlying tree of curves is isomorphic (as abstract graphs) to that of Θ. For $t_{a_i}, t_{b_j} \in \Theta$ choose curves c_i, d_j such that $t_{c_i} = \varphi(t_{a_i})$, $t_{d_j} = \varphi(t_{b_j})$. These curves may be chosen to realize their geometric intersection number.

Let M be a closed regular neighbourhood of the union of c_i and d_j for $t_{c_i}, t_{d_j} \in \varphi(\Theta)$. Note that M is an orientable surface of genus g with $n+2$ (or 3 if $n=0$) boundary components.

Similarly, let M' be a closed regular neighbourhood of the union of the curves supporting Θ. Orient M and M' so that t_{a_i}, t_{b_j} and t_{c_i}, t_{d_j} are right-handed Dehn twists. Fix an orientation preserving diffeomorphism $f_0 : M' \to M$ such that $f_0(a_{2i}) = c_{2i}$ for $1 \leq i \leq 2g-1$, $f_0(a_5) = c_5$, $\{f_0(a_1), f_0(a_3)\} = \{c_1, c_3\}$ and $\{f_0(b_j) : 1 \leq j \leq n-1\} = \{d_j : 1 \leq j \leq n-1\}$. If $(g,n) = (2,0)$ then we can also assume $f_0(a_i) = c_i$ for $i = 1,3$. Set $c_i' = f_0(a_i)$ for $i = 1,3$ and $d_j' = f_0(b_j)$ for $1 \leq j \leq n-1$. Either $(c_1', c_3') = (c_1, c_3)$ or $(c_1', c_3') = (c_3, c_1)$. Analogously, $(d_1', \ldots, d_{n-1}')$ is some (possibly nontrivial) permutation of $(d_1, \ldots, d_{n-1})$. The neighbourhood M and the curves supporting $\varphi(\Theta)$ are shown on Fig. 8.

By Lemma 3.3, $C_{\mathrm{Mod}(S)}(\varphi(\Theta)) = \varphi(C_{\mathrm{Mod}(S)}(\Theta)) = 1$. It follows that Dehn twists about the boundary components of M are trivial, hence each component of ∂M bounds either a Möbius band or a disc with 0 or 1 puncture. It is clear that exactly 1 component bounds a Möbius strip, and exactly n components bound once-punctured discs.

Consider the component u of ∂M which bounds a pair of pants together with c_1' and c_3'. By Lemma 3.5, $c_1' \cup c_3'$ can not bound a once-punctured annulus in S. It also can not bound a non-punctured annulus, because $t_{c_1} \neq t_{c_3}^{\pm 1}$. It follows that u bounds a Möbius strip.

Suppose $(g,n) \neq (2,0)$ and consider the component v of ∂M which bounds a 4-holed sphere together with c_5, c_4 and c_1'. For $i = 1,3$ set

$$x_i = (t_{a_5} t_{a_6} t_{a_4} t_{a_2} t_{a_i})^6 \quad \text{and} \quad y_i = (t_{c_5} t_{c_6} t_{c_4} t_{c_2} t_{c_i'})^6.$$

Suppose that $(c_1', c_3') = (c_3, c_1)$. Then $\varphi(x_3) = y_1$. By Lemma 2.7, x_3 is a product of 2 twists commuting with t_{a_1}, whereas y_1 does not commute with $t_{c_3'}$, a contradiction. Hence $c_i' = c_i$ for $i = 1,3$. It also follows that y_3 commutes with t_{c_1}, which implies that v bounds a non-punctured disc.

It is now clear that f_0 can be extended to $f : S \to S$. We have $\varphi(t_{a_i}) = f t_{a_i} f^{-1}$ for all $t_{a_i} \in \Theta$. Since each $t_{a_j} \in D$ can be expressed in terms of $t_{a_i} \in \Theta$, we have $\varphi(t_{a_j}) = f t_{a_j} f^{-1}$ for all $t_{a_j} \in D$. It remains to prove that $d_i' = d_i$ for $1 \leq i \leq n-1$. We proceed by induction.

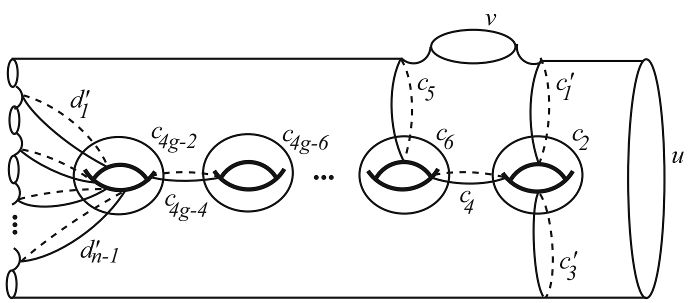


Fig. 8 The neighbourhood M supporting $\varphi(\Theta)$

Consider the once-punctured annulus A_1, whose boundary is the union of b_1 and a_{4g-3}. Let u_1 be the boundary of a small disc contained in A_1 and containing the puncture. By (a) of Lemma 2.8 we have

$$t_{u_1} = \left(t_{a_{4g-3}} t_{b_1} t_{a_{4g-2}} t_{a_{4g-4}} t_{a_{4g-6}} t_{a_{4g-7}}\right)^5 \left(t_{b_1} t_{a_{4g-2}} t_{a_{4g-4}} t_{a_{4g-6}} t_{a_{4g-7}}\right)^{-6}.$$

By applying φ to the above equality and using Lemma 2.8 we obtain that $\varphi(t_{u_1})$ is equal to a twist about the curve bounding a disc containing all punctures of the annulus A_1', whose boundary is the union of d_1 and $f(a_{4g-3})$. Since $t_{u_1} = 1$, we have $\varphi(t_{u_1}) = 1$. It follows that A_1' contains only 1 puncture, hence $d_1 = d_1'$.

Now suppose that $d_i' = d_i$ for $1 \leq i \leq k-1$ for some $k < n$. Consider the once-punctured annulus A_k, whose boundary is the union of b_{k-1} and b_k. Let u_k be the boundary of a small disc contained in A_k and containing the puncture. By (b) of Lemma 2.8 we can express t_{u_k} in terms of Dehn twists of the tree

$$\{t_{b_k}, t_{b_{k-1}}, t_{a_{4g-3}}, t_{a_{4g-2}}, t_{a_{4g-4}}, t_{a_{4g-6}}, t_{a_{4g-7}}\}.$$

By applying φ to that expression and using Lemma 2.8 we obtain that $\varphi(t_{u_k})$ is equal to a twist about the curve bounding a disc containing all punctures of the annulus A_k', whose boundary is the union of d_k and d_{k-1}. As above, it follows that $d_k = d_k'$.

For $h = 2g+2$ we proceed as above, to obtain a diffeomorphism $f_0\colon M' \to M$, where M (resp. M') is a regular neighbourhood of the union of the curves supporting $\varphi(\Lambda)$ (resp. Λ), such that $f_0(a_{2i}) = c_{2i}$ for $1 \leq i \leq 2g-1$, $f_0(a_5) = c_5$, $\{f_0(a_0), f_0(a_1), f_0(a_3)\} = \{c_0, c_1, c_3\}$ and $\{f_0(b_j)\colon 1 \leq j \leq n-1\} = \{d_j\colon 1 \leq j \leq n-1\}$, where $\varphi(t_{a_i}) = t_{c_i}$ and $\varphi(t_{b_j}) = t_{d_j}$. Set $c_i' = f_0(a_i)$ for $i = 0, 1, 3$ and $d_j' = f_0(b_j)$ for $1 \leq j \leq n-1$. Note that M and M' are orientable of genus g with $n+3$ (or 4 if $n = 0$) boundary components (see Fig. 9).

For $i \in \{0, 1, 3\}$ set

$$x_i = (t_{a_5} t_{a_6} t_{a_4} t_{a_2} t_{a_i})^6 \quad \text{and} \quad y_i = (t_{c_5} t_{c_6} t_{c_4} t_{c_2} t_{c_i'})^6.$$

Suppose $(g, n) \neq (2, 0)$ and consider the component v of ∂M which bounds a 4-holed sphere together with c_5, c_4 and c_1'. It follows from Lemma 2.7 that

$$\left\{t_{a_0}, t_{a_1}, t_{a_3}\right\} \cap C_{\mathrm{Mod}(N_h^n)}\{x_0, x_1, x_3\} = \{t_{a_1}\},$$

hence

$$\{t_{c_0}, t_{c_1}, t_{c_3}\} \cap C_{\mathrm{Mod}(N_h^n)}\{y_0, y_1, y_3\} = \{t_{c_1}\}.$$

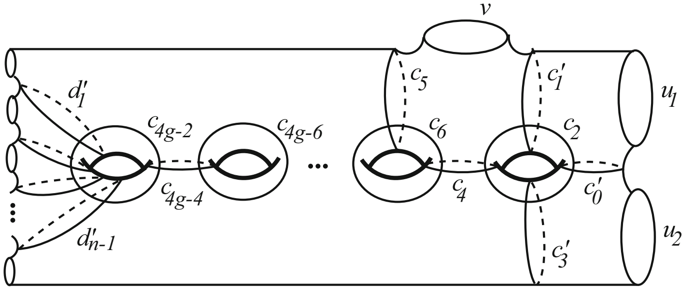


Fig. 9 The neighbourhood M supporting $\varphi(\Lambda)$

Since neither $t_{c_0'}$ nor $t_{c_3'}$ commute with y_1, we have $c_1 = c_1'$. Furthermore, since t_{c_1} commutes with y_0 and y_3, v bounds a non-punctured disc. (If $(g, n) = (2, 0)$ then $\{t_{a_0}, t_{a_1}, t_{a_3}\} \cap C_{\mathrm{Mod}(N_h^n)}\{x_0, x_1, x_3\} = \{t_{a_1}, t_{a_3}\}$. It follows that $c_0' = c_0$, and by composing f_0 by a suitable diffeomorphism if necessary we may assume $c_i' = c_i$ for $i = 1, 3$.)

By (a) of Lemma 2.8 we have

$$(t_{a_0}t_{a_1}t_{a_2}t_{a_4}t_{a_6}t_{a_5})^5(t_{a_0}t_{a_2}t_{a_4}t_{a_6}t_{a_5})^{-6} = t_a$$

For $i = 0, 3$ set

$$z_i = (t_{c_i'}t_{c_1}t_{c_2}t_{c_4}t_{c_6}t_{c_5})^5(t_{c_i'}t_{c_2}t_{c_4}t_{c_6}t_{c_5})^{-6}$$

Then either $\varphi(t_a) = z_0$ or $\varphi(t_a) = z_3$. Since t_a commutes with t_{a_3}, and z_3 does not commute with $t_{c_0'}$, we have $\varphi(t_a) = z_0$. It follows that $c_i' = c_i$ for $i \in \{0, 1, 3\}$. Note that $z_0 = t_{u_1}$, where u_1 is the component of ∂M bounding a pair of pants with c_0 and c_1. Let u_2 be the component of ∂M bounding a pair of pants with c_0 and c_3. By (b) of Lemma 2.8 we have

$$t_{u_2} = (t_{c_0}t_{c_3}t_{c_2}t_{c_4}t_{c_6}t_{c_5})^5(t_{c_3}t_{c_2}t_{c_4}t_{c_6}t_{c_5})^{-6}(t_{c_1}t_{c_3}t_{c_2})^4(t_{c_0}t_{c_1}t_{c_3}t_{c_2})^{-3}$$

By applying φ^{-1} and using (b) of Lemma 2.8 again we obtain

$$\begin{aligned}\varphi^{-1}(t_{u_2}) &= (t_{a_0}t_{a_3}t_{a_2}t_{a_4}t_{a_6}t_{a_5})^5(t_{a_3}t_{a_2}t_{a_4}t_{a_6}t_{a_5})^{-6}(t_{a_1}t_{a_3}t_{a_2})^4(t_{a_0}t_{a_1}t_{a_3}t_{a_2})^{-3}\\ &= t_a^{-1}.\end{aligned}$$

By Lemma 3.4 $C_{\mathrm{Mod}(S)}(\varphi(\Lambda)) = \varphi(C_{\mathrm{Mod}(S)}(\Lambda))$ is the infinite cyclic group generated by $\varphi(t_a) = t_{u_1} = t_{u_2}^{-1}$. It follows that $u_1 \cup u_2$ bounds an annulus (exterior to M) such that the union of M and that annulus is a nonorientable surface of genus $2g + 2 = h$.

It is clear that f_0 can be extended to $f : S \to S$. The rest of the proof follows as in the odd genus case. □

Lemma 3.6 *Let $h = 2g+1$ for $g \geq 2$, $D' = D\backslash\{t_{a_i} : i = 1, 2, 3, 4\}$ and $H = C_{\mathrm{Mod}(N_h^n)}(D')$. Let c be the nontrivial boundary component of a regular neighbourhood of the union of the curves supporting D'. Then $C_H\{t_{a_1}, t_{a_2}\}$ is the free abelian group of rank 2 generated by $(t_{a_1}t_{a_2})^3$ and either t_c if $(g, n) \neq (2, 0)$, or $(t_{a_5}t_{a_6})^3$ if $(g, n) = (2, 0)$.*

Proof Let d be the boundary of a regular neighbourhood of $a_1 \cup a_2$ (torus with one hole) and set $\rho = (t_{a_1}t_{a_2})^3$. Then $\rho^2 = t_d$. Since t_c can be expressed in terms of twists of D', we have $C_H\{t_{a_1}, t_{a_2}\} \subset C_{\mathrm{Mod}(N_h^n)}\{t_c, t_d\}$. It follows that any $x \in C_H\{t_{a_1}, t_{a_2}\}$ can be represented by a diffeomorphism, also denoted by x, equal to the identity on regular neighbourhoods of c and d. The complement of the union of such neighbourhoods has three connected components S', S'' and N, where S' is diffeomorphic to $\Sigma_{g-1,1}^n$ (containing a_5 and a_6), S'' is diffeomorphic to $\Sigma_{1,1}$ (containing a_1 and a_2), and N is diffeomorphic to $N_{1,2}$. Clearly x preserves each of these components. Furthermore, x restricts to a diffeomorphism x' of S', which commutes with all twists of D' up to isotopy. Since $\mathrm{PMod}(S')$ is generated by twists of D', $x' \in C_{\mathrm{Mod}(S')}(\mathrm{PMod}(S'))$. By [10, Proposition 5.5 and Theorem 5.6], $C_{\mathrm{Mod}(S')}(\mathrm{PMod}(S')) = C(\mathrm{Mod}(S'))$ is the infinite cyclic group generated either by t_c if $(g, n) \neq (2, 0)$, or by $(t_{a_5}t_{a_6})^3$ if $(g, n) = (2, 0)$ (note that $t_c = (t_{a_5}t_{a_6})^6$ if $(g, n) = (2, 0)$). Thus x' is isotopic on S' to some power of t_c (or $(t_{a_5}t_{a_6})^3$). Analogously, x restricts to a diffeomorphism x'' of S'', isotopic on S'' to some power of ρ. Finally, since $\mathrm{Mod}(N)$ is generated by the boundary twists, the restriction of x to N is isotopic to the product of some power of t_c and some power of t_d. □

Lemma 3.7 *Let $h = 2g + 1$ for $g \geq 2$ and suppose that φ is an automorphism of* $\mathrm{Mod}(N_h^n)$ *such that $\varphi(t) = t$ for all $t \in D$. Then φ restricts to the identity on* $\mathrm{PMod}^+(N_h^n)$.

Proof By Proposition 2.10, it suffices to prove $\varphi(u) = u$. Let $\mathcal{M}$ be the subgroup of $\mathrm{Mod}(N_h^n)$ generated by u, t_{a_1} and t_{a_2}. By [11, Theorem 4.1], $\mathcal{M}$ is isomorphic to $\mathrm{Mod}(N_{3,1})$. More specifically, it is the mapping class group of the nonorientable subsurface of N_h^n bounded by the curve c from Lemma 3.6. Set $u_2 = u$ and $u_1 = t_{a_2}^{-1} t_{a_1}^{-1} u^{-1} t_{a_1} t_{a_2}$. The following relations are satisfied in $\mathcal{M}$ (see [11]).

$$\begin{array}{ll} (1)\ t_{a_2} t_{a_1} t_{a_2} = t_{a_1} t_{a_2} t_{a_1} & (2)\ u_2 u_1 u_2 = u_1 u_2 u_1 \\ (3)\ u_2 u_1 t_{a_2} = t_{a_1} u_2 u_1 & (4)\ t_{a_2} u_1 u_2 = u_1 u_2 t_{a_1} \\ (5)\ u_i t_{a_i} u_i^{-1} = t_{a_i}^{-1} \text{ for } i = 1, 2 & (6)\ u_2 t_{a_1} t_{a_2} u_1 = t_{a_1} t_{a_2} \end{array}$$

Set $e = t_{a_2} u_2^{-1} t_{a_1} u_2 t_{a_2}^{-1}$. Note that e is a Dehn twist about the curve $t_{a_2} u_2^{-1}(a_1) = a_3$ (see (b) and (c) of Lemma 2.9). In particular $\varphi(e) = e$. Set $v = e u_1$. We have

$$e = t_{a_2} u_2^{-1} t_{a_1} u_2 t_{a_2}^{-1} \stackrel{(5)}{=} t_{a_2} u_2^{-1} t_{a_1} t_{a_2} u_2 \stackrel{(6)}{=} t_{a_2} t_{a_1} t_{a_2} u_1 u_2$$
$$v = t_{a_2} t_{a_1} t_{a_2} u_1 u_2 u_1$$

It follows from relations (1,3,4,5) that $v t_{a_i} v^{-1} = t_{a_i}^{-1}$ for $i = 1, 2$, and $v^2 = (u_1 u_2 u_1)^2 = t_c$ (for the last equality see [11, Subsection 3.2]). Observe that $v^{-1}\varphi(v)$ commutes with all twists of D', where D' is as in Lemma 3.6, and also with t_{a_i} for $i = 1, 2$. Suppose that $(g, n) \neq (2, 0)$. By Lemma 3.6, $\varphi(v) = v t_c^k (t_{a_1} t_{a_2})^{3m}$ for some $k, m \in \mathbb{Z}$. We have $t_c = \varphi(v^2) = t_c^{2k+1}$, hence $k = 0$. If $(g, n) = (2, 0)$, then by Lemma 3.6, $\varphi(v) = v(t_{a_5} t_{a_6})^{3k} (t_{a_1} t_{a_2})^{3m}$, and because $t_c = \varphi(v^2) = t_c^{k+1}$, hence $k = 0$. We have $\varphi(v) = v(t_{a_1} t_{a_2})^{3m}$. It follows that $\varphi(u_1) = u_1 (t_{a_1} t_{a_2})^{3m}$ and $\varphi(u_2) = (t_{a_1} t_{a_2})^{-3m} u_2$.

Set $t_d = (t_{a_1} t_{a_2})^6$ (d bounds a regular neighbourhood of $a_1 \cup a_2$) and $y = t_{a_2} u_2$. Observe that the curves $y(a_4)$ and a_4 are disjoint up to isotopy (recall $a_4 = c_{3,4}$), hence $y t_{a_4} y^{-1}$ commutes with t_{a_4}. By applying φ^2 we obtain that $t_d^{-m} y t_{a_4} y^{-1} t_d^m$ commutes with t_{a_4}. By [13, Proposition 4.7] it follows that $i(t_d^m(a_4), y(a_4)) = 0$. We will show that on the other hand $i(t_d^m(a_4), y(a_4)) \geq 4|m|$, which implies $m = 0$ and finishes the proof. Set $a_4' = y(a_4)$ and note that a_4', a_4 and a_1 are pairwise disjoint, and each of them intersects a_2 in a single point. We also have $i(a_4, d) = i(a_4', d) = 2$. Let M be a regular neighbourhood of $a_4' \cup a_4 \cup a_1 \cup a_2$, which is a 3-holed torus (Fig. 10). The complement of the interior of M in N_h^n is the disjoint union of a Möbius band and a subsurface diffeomorphic to $\Sigma_{g-2,2}^n$. In particular, M is an essential subsurface of N_h^n in the sense of [15, Definition 3.1], and hence, by [15, Proposition 3.3], $i(t_d^m(a_4), a_4')$ is equal to the geometric intersection number $i_M(t_d^m(a_4), a_4')$ of $t_d^m(a_4)$ and a_4' treated as curves on M. Let $\widetilde{M}$ be the 2-holed torus obtained from M by gluing a disc along the boundary component f (see Fig. 10). Clearly $i_M(t_d^m(a_4), a_4') \geq i_{\widetilde{M}}(t_d^m(a_4), a_4')$, and since a_4' is isotopic on $\widetilde{M}$ to a_4, we have $i_{\widetilde{M}}(t_d^m(a_4), a_4') = i_{\widetilde{M}}(t_d^m(a_4), a_4)$. Finally, by [10, Proposition 3.3] $i_{\widetilde{M}}(t_d^m(a_4), a_4) = |m| i_{\widetilde{M}}(d, a_4)^2 = 4|m|$. Summarising, we have

$$i(t_d^m(a_4), a_4') = i_M(t_d^m(a_4), a_4') \geq i_{\widetilde{M}}(t_d^m(a_4), a_4') = 4|m|$$

□

Lemma 3.8 *Let $h = 2g + 2$ for $g \geq 2$ and suppose that φ is an automorphism of* $\mathrm{Mod}(N_h^n)$ *such that $\varphi(t) = t$ for all $t \in E \cup \{t_a\}$. Then φ restricts to an inner automorphism of* $\mathrm{PMod}^+(N_h^n)$.

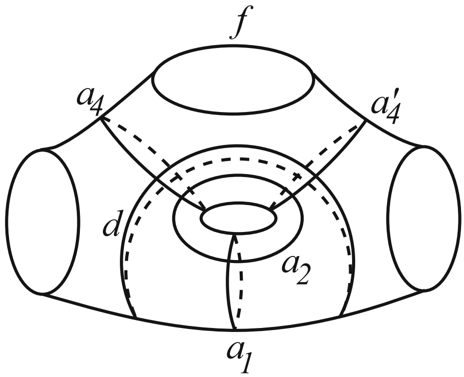


Fig. 10 The regular neighbourhood M of $a_4' \cup a_4 \cup a_1 \cup a_2$

Proof Let K denote the nonorientable connected component of the surface obtained by removing from N_h^n an open regular neighbourhood of $a_1 \cup a_3$. Thus, K is a Klein bottle with two holes, and the other connected component is diffeomorphic to $\Sigma_{g-1,2}^n$. Furthermore, by (e) of Lemma 2.9, K is a regular neighbourhood of $c_{1,6} \cup c_{5,6} \cup c_{6,6}$. Using [15, Corollary 3.8] we will treat $\mathrm{Mod}(K)$ as a subgroup of $\mathrm{Mod}(N_h^n)$.

Set $u' = \varphi(u)$. Since u' commutes with all twists of E supported on $N_h^n \backslash K$, it can be represented by a diffeomorphism supported on K, by a similar argument as in the proof of Lemma 3.6. Hence $u' \in \mathrm{Mod}(K)$. Since $u' t_{a_0} u'^{-1} = t_{a_0}^{-1}$, u' preserves the isotopy class of a_0 by [13, Proposition 4.6]. Let $\mathcal{S}$ denote the subgroup of $\mathrm{Mod}(K)$ consisting of elements fixing the isotopy class of a_0, and let $\mathcal{S}^+$ be the subgroup of index 2 of $\mathcal{S}$ consisting of elements preserving orientation of a regular neighbourhood of a_0. Note that every element of $\mathcal{S}^+$ can be represented by a diffeomorphism equal to the identity on a neighbourhood of a_0. By cutting K along a_0 we obtain a four-holed sphere, and it follows from the structure of the mapping class group of this surface, that $\mathcal{S}^+$ is isomorphic to $\mathbb{Z}^3 \times F_2$, where the factor $\mathbb{Z}^3$ is generated by t_{a_1}, t_{a_3} and t_{a_0}, and F_2 is the free group of rank 2 generated by t_a and $u t_a u^{-1}$.

Set $v = t_a u$. By [12, Lemma 7.8] we have $v^2 = t_{a_1} t_{a_3}$. Note that $v \in \mathcal{S} \backslash \mathcal{S}^+$. It follows from the previous paragraph, that $\mathcal{S}$ admits a presentation with generators t_{a_1}, t_{a_0}, t_a and v, and the defining relations

$$t_{a_0} t_a = t_a t_{a_0}, \qquad v t_{a_0} = t_{a_0}^{-1} v, \qquad v^2 t_a = t_a v^2$$
$$t_{a_1} v = v t_{a_1}, \qquad t_{a_1} t_{a_0} = t_{a_0} t_{a_1}, \qquad t_{a_1} t_a = t_a t_{a_1}$$

Let H denote the subgroup generated by t_{a_0}, t_{a_1} and $v^2 = t_{a_1} t_{a_3}$. It follows from above presentation that H is normal in $\mathcal{S}$ and $\mathcal{S}/H$ is isomorphic to the free product $\mathbb{Z} * \mathbb{Z}_2$. More specifically, denoting by A and V the images in $\mathcal{S}/H$ of respectively t_a and v, we see that $\mathcal{S}/H$ has the presentation $\langle A, V \mid V^2 = 1 \rangle$.

Since $\varphi(t_{a_i}) = t_{a_i}$ for $i = 0, 1, 3$ and $\varphi(t_a) = t_a$ and $u' = \varphi(u) \in \mathcal{S}$, φ preserves $\mathcal{S}$ and, by the same argument, φ^{-1} also preserves $\mathcal{S}$, hence $\varphi|_{\mathcal{S}}$ is an automorphism of $\mathcal{S}$. Since φ is equal to the identity on H, it induces $\phi \in \mathrm{Aut}(\mathcal{S}/H)$. We have $\phi(A) = A$. Note that every element of order 2 in $\mathcal{S}/H$ is conjugate to V. In particular $\phi(V)$ is conjugate to V. It is an easy exercise to check, using the normal form of elements of the free product, that in order for ϕ to be surjective, we must have $\phi(V) = A^n V A^{-n}$ for some $n \in \mathbb{Z}$.

It follows that $\varphi(v) = t_{a_1}^k t_{a_3}^l t_{a_0}^m t_a^n v t_a^{-n}$ for some integers l, k and m. We have $t_{a_1} t_{a_3} = \varphi(v^2) = t_{a_1}^{2k+1} t_{a_3}^{2l+1}$, hence $k = l = 0$. By composing φ with the inner automorphism $x \mapsto t_a^{-n} x t_a^n$ we may assume $n = 0$ (note that t_a commutes with all t_{a_i}). Thus $\varphi(u) = t_{a_0}^m u$.

Set $y = t_{a_0} u$ and note that $y(a_2)$ is disjoint from a_2, hence $y t_{a_2} y^{-1}$ commutes with t_{a_2}. By applying φ we obtain that $t_{a_0}^m y t_{a_2} y^{-1} t_{a_0}^{-m}$ commutes with t_{a_2}, which gives $i(t_{a_0}^{-m}(a_2), y(a_2)) = 0$. On the other hand, by a similar argument as in the proof of Lemma 3.7, we have $i(t_{a_0}^{-m}(a_2), y(a_2)) \geq |m|$, hence $m = 0$. □

Proof of Theorem 1.1 By Lemma 3.1 we can assume $S_1 = S_2$. Suppose that φ is any automorphism of $\mathrm{Mod}(N_h^n)$ for $h \geq 5$. By Lemma 3.2, there exists $f \in \mathrm{Mod}(N_h^n)$ such that φ' defined by $\varphi'(x) = f^{-1}\varphi(x) f$ for $x \in \mathrm{Mod}(N_h^n)$ is the identity on D (if h is odd) or $E \cup \{t_a\}$ (if h is even). By Lemma 3.7 or Lemma 3.8, φ' restricts to an inner automorphism of $\mathrm{PMod}^+(N_h^n)$. Thus, by composing φ' with an inner automorphism we obtain φ'', which restricts to the identity on $\mathrm{PMod}^+(N_h^n)$. Since $C_{\mathrm{Mod}(N_h^n)}(\mathrm{PMod}^+(N_h^n))$ is contained in $C_{\mathrm{Mod}(N_h^n)}(\mathcal{T}(N_h^n))$, it is trivial by [13, Theorem 6.2]. Lemma 1.3 implies that φ'' is trivial, hence φ is inner. □

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